# Enhancing the mathematical interpretation of the Root-Locus method of closed-loop control system analysis Using MATLAB

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## Abstract

A closed-loop control system that has an initial transient response can be stable or unstable depending on what happens after the initial transient response. If the initial transient response continues in the unstable direction, the control system is not usable for the input parameters because it is unstable. On the other hand, if the initial transient response is countered by the control system action, the system becomes stable, and the control system is usable for the input parameters. The basic characteristics of the transient response of a closed-loop control system can be determined based on the location of its poles on its Root-Locus plot. How the poles move and their mathematical interpretation in time domain are predictors of the control system performance. Addition of a compensator is a technique that can be used to make the necessary adjustments to stabilize an unstable model. In this article, MATLAB is used to analyze the stability of a control system and to present the concept of adding a compensator to a control system. Additionally, use of MATLAB for getting a closed-loop control system response in time domain is discussed. Studying the system response in time domain enhances the understanding of the Root-Locus technique.

**Keywords:**
MATLAB, Root-Locus, Controls, Stability

## Technical discussion

In control system engineering, parametric studies of the system variables are common. The goal of parametric studies is to predict system stability characteristics. Figure 1 is a graphical representation for a single degree of freedom (SDF) vibrating system and figure 2 is the block diagram of a closed loop control system.[1]

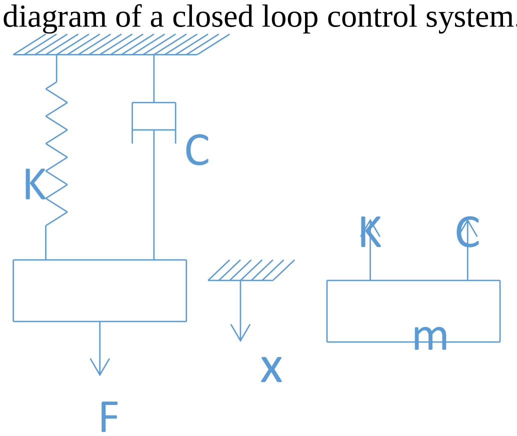


Figure 1: SDF vibrating system

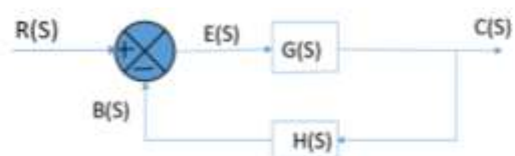


Figure 2: Block diagram of a closed loop control system

Differential equation of a SDF vibrating system with external excitation is shown in equation 1.

$m\, d^2X / dt^2 + C\, dX / dt + KX = F$ (equation 1)

The transfer function of equation 1 is defined by equation 2.

Transfer function = $1 / (m S^2 + C S + K)$ (equation 2)

The roots of the denominator of equation 2 are shown in equation 3.

$S= -C \pm(C^2 - 4mK)^{0.5}/ 2m$ (equation 3)

There are 3 possible scenarios from a stability point of view for the solution of equation 3. Equation 3 can have only negative real roots, at least one positive root, or 2 complex roots consisting of a real part and an imaginary part.

Formulation of the vibrating system of figure 1 cannot always result in all three 3 scenarios for equation 3 because for a single degree of freedom vibration system, damping cannot be negative. However, a modal analysis of a complex structure can result in various forms of equation 3 that would produce equations for certain modes that could fall under any of the 3 scenarios.[2]

Equation 4 is an example of scenario 1 (2 negative roots), equation 5 is an example of scenario 2 (at least one positive root) and equation 6 is an example of scenario 3 (complex roots).

$X(S) = 1 / (S^2 + 3 S + 2)$ (equation 4)
$X(S) = 1 / (S^2 - 3 S + 2)$ (equation 5)
$X(S) = 1 / (S^2 + 2 S + 2)$ (equation 6)

The inverse Laplace Transform of equations 4 through 6 are their time responses. Equations 7, 8 and 9 are the inverse Laplace Transform of equations 4, 5 and 6. Equations 7 through 9 can be obtained by manual techniques or by using MATLAB.

$X(t) = e^{-t} - e^{-2t}$ (equation 7)
$X(t) = -e^{t} + e^{2t}$ (equation 8)
$X(t) = e^{-t}\sin t$ (equation 9)

Both components of equation 7 die down as time is increased. Therefore, it is concluded that equation 4 represents a stable system. Both components of equation 8 increase as time is increased. Therefore, it is concluded that equation 5 represents an unstable system. Equation 9 dies down as time is increased. Therefore, equation 6 represents a stable system. An equation with complex roots can also be unstable. For example if the “$e^{-t}$” term in equation 9 were “$e^{t}$”, the system would be unstable

The Root-Locus plot for equations 4, 5 and 6 can be plotted manually which is time consuming or quickly in MATLAB. All the points to the left of the imaginary axis of a Root-Locus plot represent the stable portion of the response, and all the points to the right represent the unstable portion of the response.

Figure 3 is the Root-Locus plot of equation 4. Figure 4 is the Root-Locus plot of equation 5. Figure 5 is the Root-Locus plot of equation 6. All the Root-Locus plots are plotted using MATLAB[3]. The Root-Locus plots are confirming the stability conclusions reached from equations 7, 8 & 9.

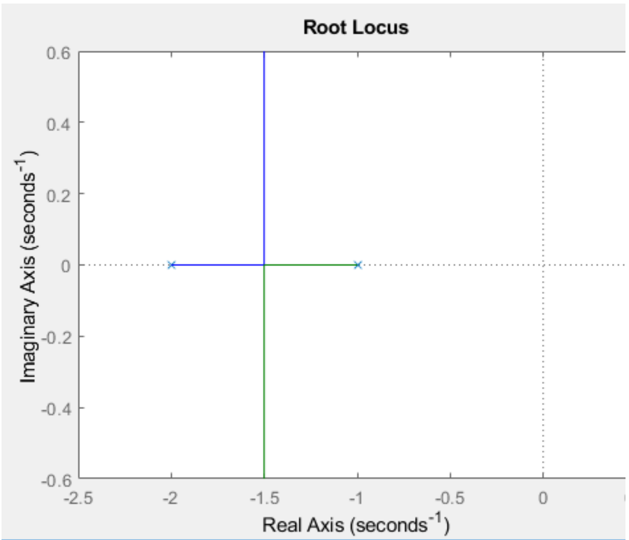


Figure 3: Root-locus plot of equation 4

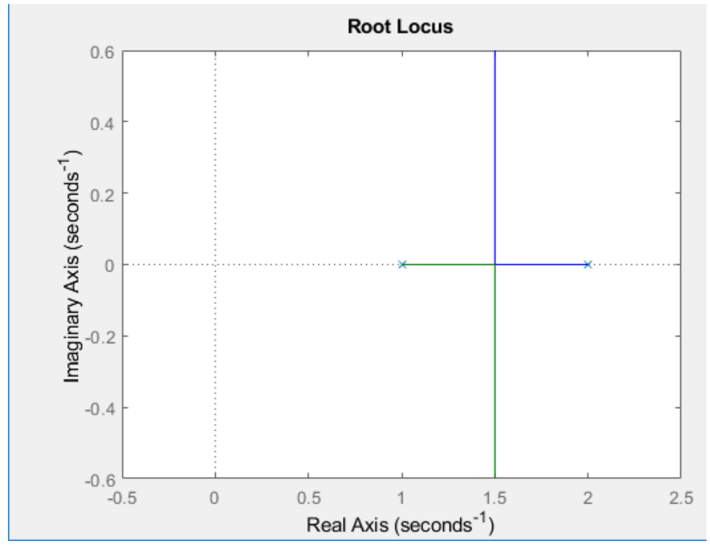


Figure 4: Root-locus plot of equation 5

Equations 4, 5 and 6 were all developed based on a vibrating system which is an open loop control system. Figure 2 is an illustration of a closed loop control system. The output divided by the input of the closed loop control system of figure 2 is defined by equation 10.[4] Equation 10 puts a closed loop control system in a format similar to an open loop control system where open loop stability analysis techniques can be applied.

$C(S)/ R(S) = G(S) / \{1 + G(S)H(S)\}$ (equation 10)

From equation 10, the transfer function of equation 11 is obtained which is in the same format as equation 2.

Transfer function = $G(S) / \{1 + G(S)H(S)\}$ (equation 11)

The similarity between equations 11 and 2 shows that the Root-Locus techniques and the Inverse Laplace Transform techniques demonstrated using variations of equation 2 apply to closed loop control systems. Consider a scenario of a closed loop control system of the type shown in figure 2. Assume the transfer function of the closed loop system of equation 11 is defined by equation 12.

$C(S) = (S^3 + 1.41) / (S^4 + 3\ S^2 + 6)$ (Equation 12)

MATLAB can be used to put equation 12 into the format shown in equation 13.

$X(S) = \{(0.1456 + 0.1424i)/(S - 0.6890 – 1.4053i)\} + \{(0.1456 – 0.1424i) / (S – 0.6890 + 1.4053i)\} + \{(0.3544 – 0.2449i)/(S + 0.6890 – 1.4053i)\} + (0.3544 + 0.2449i)/(S + 0.6890 + 1.4053i)\}$ (equation 13)

Equation 13 can be mathematically manipulated to result in equation 14.

$X(S) = \{(0.29S – 0.4)/(S-0.69)^2 + 1.4^2\} + \{(0.71S + 1.79)/(S+0.69)^2 + 1.4^2\}$ (equation 14)

Equation 15 is the Inverse Laplace Transform of equation 14 which can be obtained manually or by using MATLAB.

$X(t) = e^{0.69t} (-0.42 \sin 1.4t + 0.2 \cos 1.4t) + e^{-0.69t} (0.92 \sin 1.4t + 0.71 \cos 1.4t)$ (equation 15)

Standard trigonometric relationships can be used to put equation 15 in the form of equation 16.

$X(t) = 0.46\, e^{0.69t}\sin(1.4t – 2.67) + 1.16\, e^{-0.69t}\sin(1.4t+0.785)$ (equation 16)

The "$e^{0.69t}\sin(1.4t – 2.67)$" component of equation 16 shows that the system represented by equation 16 is unstable because "$e^{0.69t}$" grows as time increases.

Figure 6 shows the root locus plot of equation 12. Equation 15 shows that the function shown in equation 12 has 2 stable components and 2 unstable components. The Root-Locus plot of figure 6 confirm these results. The root-locus plot of figure 6 can be used to determine the necessary adjustments required to make the system stable. This describes the concept of designing and adding a compensator to the system. Space limitation in this article does not allow for a detailed presentation and example of a compensator design.

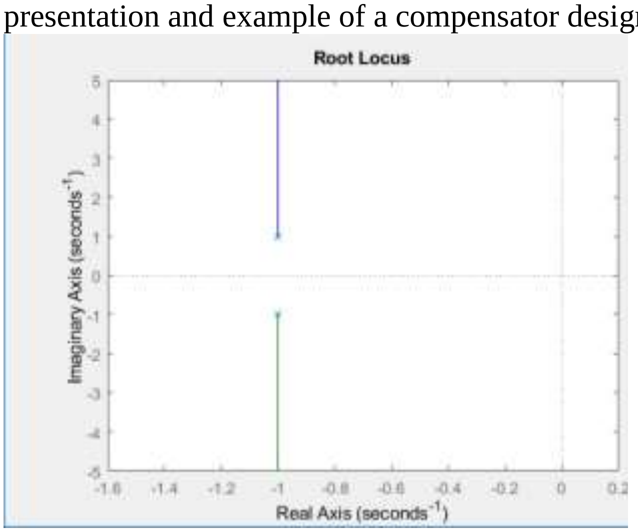


Figure 5: Root-locus plot of equation 6

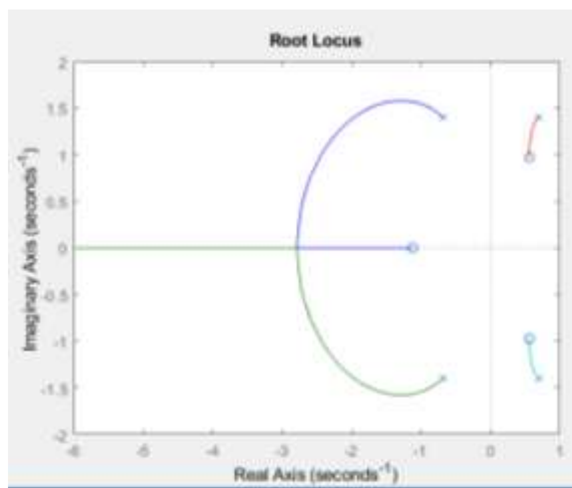


Figure 6: Root-locus plot of equation 12

**Summary & Conclusion:**

It has been shown that the Root-locus method can be used to determine the stability status of both open loop and closed loop control systems. Root-Locus plots and time domain responses can be quickly obtained by using MATLAB. Comparison of the Root-Locus plots and time domain responses of the control system enhance and clarify the conclusions reached regarding system stability, and they can guide the design and implementation of compensators.

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## Biographical Information

Dr. Hagigat is an associate professor in the college of engineering of University of Toledo, and he is teaching engineering technology courses. Dr. Hagigat has an extensive industrial background, and he is continuously emphasizing the practical applications of engineering material covered in a typical engineering technology course.