Basic Quantum Concepts for Engineering Undergraduates: Making More Effective Use of Heisenberg's Uncertainty Principle

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Abstract – This paper deals, in part, with the difficulties faced by engineering instructors when introducing quantum concepts into an undergraduate engineering curriculum. The approach taken by the present author is to try to highlight connections between the seemingly abstract concepts and the more practical engineering world. To do this, the paper focuses on a specific quantum topic, namely the Heisenberg Uncertainty Principle, often viewed as a somewhat archaic curiosity. The latter is shown, instead, to be the basis of a viable approach to explaining phenomena as diverse as the electrical resistance of nanowires, the quantization of magnetic flux and of magnetic moment. At a more senior undergraduate level, the same approach is shown to provide a reasonable explanation for the phenomenon of quantum linear magnetoresistance, observed experimentally in semiconductors in the cryogenic region.

Keywords: uncertainty principle; quantum conductance; quantum linear magnetoresistance

INTRODUCTION

The advent of nanosized electronic devices and memory storage elements has meant that engineers increasingly have to deal with issues best explained in quantum rather than classical terms. When quantum concepts are introduced in junior level engineering courses, however, significant challenges await the instructor in terms of sustaining student interest in and overall satisfaction with the material. Some of those can rightly be attributed to the abstruse nature of various topics as well as to typical engineering students being outside of their comfort zone. The educational challenge is, therefore, to demonstrate that some of the topics covered are more than simply curious relics of early 20th century scientific breakthrough but, rather, are part of a useful framework for understanding basic nanoscale phenomena. That argument is made in this paper by focusing on one specific topic, namely, Heisenberg's Uncertainty Principle (HUP). As far as it is possible to tell, the most typical gesture by instructors to the world of engineering when covering this topic is that of drawing parallels between HUP and Fourier transforms. In this paper, the engineering slant is taken further by focusing on demonstrating the simplicity with which the above-mentioned Principle can be applied in areas of engineering interest. A prime example of this is using HUP to arrive at a correct formulation for the resistance of metallic nanowires. Other simple examples described, here, include quantization of magnetic moment and of magnetic flux, both of which are relevant to issues concerning nanoscale magnetic systems such as cutting-edge hard disk storage. Finally, a more adventurous application of the Principle, quantum linear magnetoresistance, a cryogenic phenomenon with considerable long-term device potential is described for the first time in outline.

HEISENBERG'S UNCERTAINTY PRINCIPLE

Exposure to the Heisenberg Uncertainty Principle is almost a rite of passage for science and engineering students in undergraduate courses concerned primarily with an introduction to quantum concepts. The principle is embodied in two simple algebraic relationships: $\Delta P \Delta x \geq h$ or $\Delta E \Delta t \geq h$ where $\Delta P, \Delta x, \Delta E, and \Delta t$ refer, respectively, to 'uncertainties' in the momentum, position, energy and time of a fundamental entity such as an electron or a photon and h is Planck's constant. When

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dealt with, the salient points of discussion generally center on emphasis of the futility of trying to obtain precise simultaneous experimental determinations of (say) elementary particle momentum and location, or of particle energy at a precisely defined time [1].

During discussion of the Principle, a little extra 'mileage' may be obtained from a demonstration of its consistency with the rationale of the Fourier Transform. In such an exercise, the uncertainty in the temporal parameter can be identified with the width of (say) a rectangular pulse in the time domain. The corresponding uncertainty in frequency, and consequently, energy can be obtained from the first crossover of the sine function of the corresponding transform. This example though of interest, particularly for engineering students, is not a significant demonstration of the usefulness of the Principle as applied to topics in the sphere of quantum concepts.

The substantial recent growth of interest in nanotechnology and in the associated physics of nanoscopic and mesoscopic systems and devices appears to provide some opportunity to re-evaluate the Principle as a useful tool in the estimation of certain basic quantized parameters in a manner relatively easily assimilated by science and engineering undergraduates. This is particularly true if the quantity being derived in question can, formally, only be arrived at after tortuous calculation.

Heisenberg's findings have, of course, been re-examined by true specialists over decades and revised forms of the Principle have been published on various occasions [1]. These modified forms have been eschewed here and the original formulation is applied.

In the following, three topics are briefly outlined as a means of illustrating the above argument. These include the quantized nature of electrical resistance in nanowires as well as the quantization of magnetic flux and of magnetic moment. The latter two are outlined in Appendices A and B.

Estimation of the Quantum Electrical Resistance of a Metallic Nanowire

Quantum conductance in a metallic nanowire of near-atomic lateral dimensions clearly involves fluctuations and, thereby, fluctuations in numbers of fundamental charge carriers.

Applying Ohm's Law, it follows that

$$\Delta R = \Delta V / \Delta I \qquad (1)$$

where the symbols, R, V, I, have their usual meaning and $\Delta V, \Delta I$, represent uncertainties in those parameters.

Since electronic charge, e, electrostatic potential, V, and energy, E, are related via $eV = E$, it follows that

$$\Delta V = \Delta E / e. \qquad (2)$$

Also, in general, $\qquad \Delta I = \Delta q / \Delta t, \qquad (3)$

where Δq represents the uncertainty in charge passed in a time span of uncertainty Δt. Clearly, this uncertainty in charge may be identified with the electronic charge, e. Accordingly, Eq.(3) becomes

$$\Delta I = e / \Delta t \qquad (4)$$

It follows from Eqs.(1), (2) and (4), therefore, that $\Delta R \geq (\Delta E \Delta t)/e^2$

Applying the HUP, results in $\Delta R \geq h/e^2$.

The ubiquitous factor of two from electron spin also, obviously, comes into play. This is effected here by treating the quantum resistance, R_q, of a nanowire as a random variable with spin accounted for through an associated (symmetric) probability density function, $f_R(R)$, the latter being assumed to span the range $0 \rightarrow \Delta R$. The expectation for the nanowire resistance, $E[R_q]$, is clearly then,

$$E[R_q] = \Delta R/2 = h/2e^2 \quad (5)$$

in full agreement with the conventional result, namely one half of the von Klitzing constant [2].

Quantization of Magnetic Flux

The result of Eq.(5) is not model-specific and relies only upon basic definitions.

The quantization of magnetic flux, φ, as encountered in mesoscopic superconducting systems and devices [3] is dealt with, here, in similar fashion to the above, starting simply with the integration of flux density, **B**, over an enclosed cross-sectional area normal to the field:

i.e. $\qquad \varphi = \iint \mathbf{B.dS},$

where, in nanosized systems, Eq.(6) reduces, simply, to

$$\varphi = \mathbf{B.S} \qquad\qquad (6)$$

and results in the expectation for the quantum of magnetic flux, $E[\varphi_q]$, being expressed as

$$E[\varphi_q] = h/2e \qquad\qquad (7)$$

The derivation of Eq.(7) is outlined in Appendix A.

Quantization of Orbital Magnetic Moment

The spectacular increase in bit storage density in magnetic hard drive systems means that the magnetic moments of storage elements are rapidly approaching the quantum limit. It is incumbent upon engineers to familiarize themselves with the notion of magnetic moment being ultimately quantized in nature. One benefit of this particular calculation is that students generally will have already encountered the Bohr magneton as a corollary to the latter's model of the hydrogen atom.

In Appendix B, a similar approach to that of Appendix A has been adopted in arriving at an expression for the quantization of orbital magnetic moment. In neither instance is there any reliance on specific models. The resulting expressions are arrived at simply by using basic definitions and by a similar application of Heisenberg's Principle.

Quantum Hall Effect and Quantum Linear Magnetoresistance

The linear field dependence of magneto-resistance in a variety of metals has, for decades, been a topic of considerable and sustained scientific interest [4-11]. A considerable body of literature appears to favor explanations based upon intrinsic properties of these materials. That body, itself, sub-divides further into two sets, one focusing on common, but atypical, metallic band features such as (strong) spin correlations, (small) energy gaps and ground states defined by spin density waves [8,9]. The other set concentrates on localized anomalies in the Fermi surface leading to 'hot spots' [10] and related issues [11]. All of these models are convincing, but, for the most part, fairly complex. The non-specialist may, therefore, be left to wonder whether the observed transition from the 'normal' quadratic MR effect into a linear form, below the quantum limit, may be explained more simply, but, at the same time, in a form consistent with the existing body of knowledge.

In recent years interest in linear magneto-resistance (LMR) has escalated further through the advent of published data of MR ratios of simply enormous magnitude in certain doped semiconductors [12-14]. An additional key feature of the effect in those materials is the apparent absence of saturation even in massive applied fields, in sharp contrast to the established form for quadratic formulations of the effect [15]. The accompanying literature has focused, once again, on arriving at either quantum mechanical [16] or classical explanations of the phenomenon [17].

In contrast to the complexity of some of the above-mentioned models and their accompanying calculations, the approach taken in the present work is unorthodox but strikingly simple. The method involves assessing the uncertainty, below the quantum limit, in an expression for resistivity derived from diagonal elements of a general classical skew-symmetric conductivity tensor (Appendix A). This leads to fundamental fluctuations in a part of the resistivity, quadratic in magnetic field, normally associated with conventional MR. At an appropriate point, Heisenberg's uncertainty principle is invoked, leading to a simple expression for LMR that appears to agree with published experimental work [12, 14]. An arguably similar approach has already been used successfully in the calculation of ballistic conductance in nanowires [15].

It should be noted that the conductivity tensor in question is based solely on considerations of intrinsic symmetry [14], meaning that morphological questions are not dealt with here. That does not, of course, imply that the latter are unimportant; quite the opposite, in fact. Both first and second-order galvanomagnetic effects can be shown to be crucially dependent on crystal symmetry [19]. Such considerations, however, are beyond the scope of this study and are not considered further.

Outline of the Method

The analysis summarized in Appendix A results in a general skew-symmetric conductivity tensor (see (7A)) with the leading diagonal elements, σ_{xx}, σ_{yy}, given, at zero frequency, by

$$\sigma_{xx} = \sigma_{yy} = \frac{\varepsilon_o \omega_\rho^2 \omega_R}{\left(\omega_R^2 + \omega_c^2\right)} \tag{8}$$

where $\omega_p = \left[\dfrac{ne^2}{\varepsilon_o m^*}\right]^{1/2}$ is the plasma frequency,

ω_R is defined, here, as $\dfrac{2\pi}{\tau}$, where τ is a relaxation time dominated by cyclotron scattering,

$$\omega_c = \frac{eB}{m^*}$$

is the cyclotron frequency, B is the applied magnetic field and m^* is the effective mass of the carrier. Inversion of (1), using tensor (7A) allows a determination of the leading diagonal element, ρ_{xx}, of the resistivity tensor.

i.e.
$$\rho_{xx} = \frac{\sigma_{xx}}{\sigma_{xx}^2 + \sigma_{xy}^2} \tag{9}$$

When (1) and (8C) are substituted into (2), however, second order field terms disappear (for all diagonal elements of the $[\rho]$ tensor). This issue can be dealt with in two ways. First, if the condition $\left(\dfrac{\omega_c}{\omega_R}\right)^2 \langle\langle 1$, (i.e. $\omega_c \tau \leq 1$), is applied, σ_{xy} may be safely ignored and $\rho_{xx} \cong \dfrac{1}{\sigma_{xx}}$.

Further, it can be shown that if Ziman's open orbit analysis [12] is applied in, say, the x- direction, then a result similar to (2) is obtained. When the above restriction on $\omega_c \tau$ is applied, then, once again, diagonal elements of the resistivity and conductivity tensors have a simple reciprocal relationship. Thus, the diagonal resistivity, ρ_{xx}, is now

$$\rho_{xx} = \frac{\left(\omega_R^2 + \omega_c^2\right)}{\varepsilon_o \omega_p^2 \omega_R} = \frac{\omega_R}{\varepsilon_o \omega_p^2} + \frac{\omega_c^2}{\varepsilon_o \omega_p^2 \omega_R} = \rho + \rho_{xMR} \tag{10}$$

It is important to note, here, that
- (i) ρ is not the zero field resistivity. It is, actually, the resistivity of the sample when the applied field is in the z-direction (i.e. $\rho = \rho_{zz}$).
- (ii)   The zero field resistivity, ρ_o, arises from the solution of (1A) for the case $B_z = 0$. In fact

$$\rho_o = \frac{\omega_o}{\varepsilon_o \omega_p^{\,2}} \qquad (11)$$

where $\omega_o \left(= \dfrac{1}{\tau_o} \right)$ is the carrier relaxation frequency in zero field. It should be emphasized here that $\omega_c \tau_o$ is without restriction.

For a fixed (z-direction, say) of current flow, the application of orthogonal fields between the x- and z- directions results in the quadratic field-dependent resistivity differential, ρ_{xMR}.

LMR may be viewed as a fundamental fluctuation in this quantity in the quantum limit (i.e. at temperatures T $\langle \dfrac{\hbar \omega_c}{k_B}$) [15]. The resulting uncertainty in ρ_{xMR}, labeled here as $\Delta \rho_x$, can, therefore be shown to be

$$\Delta \rho_x = \Delta \left(\frac{\omega_c^{\,2}}{\varepsilon_o \omega_p^{\,2} \omega_R} \right) \qquad (12)$$

where the Δ prefix on the right side of (12) refers to the uncertainty in that expression. It follows that

$$\Delta \rho_x = \frac{2|\omega_c|(\Delta \omega_c)}{\varepsilon_o \omega_p^{\,2} (\Delta \omega_R)} \qquad (13)$$

The modulus sign, bracketing ω_c , simply means that the magnitude of the uncertainty is independent of the polarity of the z-directed magnetic field, B.

If the right side of Eq. (13) is multiplied, top and bottom, by $\hbar$, then

$$\Delta \rho_x = \frac{2|\omega_c|(\hbar \Delta \omega_c)}{\varepsilon_o \omega_p^{\,2} (\Delta \omega_R) \hbar} = \frac{|\omega_c|(\hbar \Delta \omega_c)(\Delta \tau)}{\pi \varepsilon_o \omega_p^{\,2} \hbar} \qquad (14)$$

It is generally accepted that, for all but the very highest fields, only the first Landau level is attainable by carriers in the quantum region [9]. The quantity $\hbar \Delta \omega_c \left(= \Delta E \right)$ may, therefore, be viewed as an uncertainty in the energy gap between the lowest Landau level and the ground state energy caused by (say) spin-density or charge-density fluctuations in the latter [8,9]. Likewise, $\Delta \tau$ represents uncertainty in the precessional scattering relaxation time. Equation (14) may, therefore, be re-expressed as

$$\Delta \rho_x = \frac{|\omega_c|(\Delta E \Delta \tau)}{\pi \varepsilon_o \omega_p^{\,2} \hbar}, \qquad (15)$$

Here, $\Delta \rho_x$ is treated as a random variable, of lower limit zero, with a symmetric probability density function based upon spin up/spin down considerations [15], meaning that its expectation, $E[\Delta \rho_x](= \Delta \rho)$, is $\dfrac{\Delta \rho_x}{2}$.

Applying the Heisenberg's uncertainty principle, in the usual manner and combining (11) and (15) results in a magnetoresistance ratio of the form

$$\frac{\Delta \rho}{\rho_o} = \tau_o |\omega_c| = \left(\frac{e \tau_o}{m^*} \right) |B| = \mu_c |B|, \qquad (16)$$

where μ_c, $(= \mu_c(T))$, is the carrier mobility. This equation is identical to an empirical relationship fashioned by Johnson et al [14] from LMR data on MnAs-GaAs composites and one which appears to provide an explicit form for Kohler's Rule [15] in doped semiconducting systems.

Equation (16) can be assessed in a number of ways, as shown in the following.

First, it shows that the LMR ratio/unit applied field, for certain semiconductor systems, depends only on carrier mobility and, as such, is in good agreement with published work [11]. It also implies that Kohler's constant and the Hall coefficient are one and the same.

It is also instructive to apply (9) to the experimental data of Hu and Rosenbaum [9] on samples of doped InSb. One specific result, for example, cited in that paper, is a $\dfrac{\Delta\rho}{\rho_o}$ value of 400 at a temperature of 50K and with an applied field of 13 T. Using their quoted figures for sample resistivity (~50 times that of Cu), and carrier density (5 x 10^{17}/cm^3), results in a value of $\dfrac{\tau_o}{m^*}$ of 1.46 x 10^{20} s/kg, resulting in $\dfrac{\Delta\rho}{\rho_o} = 304$, which is within 24% of the measured value, indicating that the LMR in that system appears to be, for the most part, quantum mechanical in origin. The calculated temperature dependence of $\dfrac{\Delta\rho}{\rho_o}$ in InSb over the range 30K to 175K is also reasonably compliant with their data.

It is a simple matter to express (9) in the form $\dfrac{\Delta\rho}{\rho_o} = S\left(\omega_c\tau\right)$ similar to that used by Falicov and Smith [8] where S is a dimensionless slope parameter. In the present analysis, S is a dimensionless quantity of the form $\dfrac{\tau_o}{\tau}$.

i.e.
$$\frac{\Delta\rho}{\rho_o} = \left(\frac{\tau_o}{\tau}\right)\left(\omega_c\tau\right) \tag{17}$$

It may also be of interest to compare Eq. (17) with Abrikosov's [5] original description of LMR in metals. His expression, (in SI units) is

$$\Delta\rho = \left(\frac{N_i}{\pi\, n^2 e}\right)\left|B\right| \quad .$$

Here, N_i is the number density of static scattering centers and n is the electron density. A little manipulation produces an expression of the form

$$\frac{\Delta\rho}{\rho_o} = \left(\frac{1}{\pi}\right)\left(\frac{\tau_o}{\tau}\right)\left(\frac{N_i}{n}\right)\left(\omega_c\tau\right). \tag{18}$$

In doped semiconductors, where N_i and n tend to converge, (10) and (11) end up looking fairly similar.

It may also be shown that, although incidental to this paper, the same basic approach may be used with the off-diagonal Hall conductivity, σ_{xy}, (8C), for, say, a two-dimensional mesoscopic sample. The off-diagonal resistivity, ρ_{xy}, can easily be shown, in the classical region, to be

$$\left|\rho_{xy}\right| = \frac{\omega_c}{\varepsilon_o\omega_p{}^2} = \left(\frac{\omega_s}{\varepsilon_o\omega_p{}^2}\right)\left(\frac{\omega_c}{\omega_s}\right)$$

i.e.
$$\frac{\rho_{xy}}{\rho_o} = \frac{\omega_c}{\omega_s},$$

where ω_s is a relaxation frequency limited, among other things, by surface scattering and ρ_o is the resistivity of the mesoscopic sample in zero field.

Addressing the uncertainty in this quantity below the quantum limit, with ω_s equated to $\dfrac{2\pi}{\tau_s}$, results in

$$\Delta\left(\frac{\rho_{xy}}{\rho_o}\right) = \frac{\Delta\omega_c}{\Delta\omega_s} = \frac{(\hbar\Delta\omega_c)(\Delta\tau_s)}{2\pi\hbar} \cong 1.$$

Once again, application of Heisenberg's uncertainty principle suggests that, in the quantum region, the uncertainty in Hall resistance is the same as that of the resistance of the sample, itself, (along the axis of the Hall voltage). This, in turn, results, as expected, in the von Klitzing constant [2,18].

In conclusion, this analysis, despite its inherent simplicity, does not appear to be in serious conflict with established work in this field. When applied to published experimental data, there seems to be good agreement. Equation (16) also clearly demonstrates the crucial nature of carrier mobility below the quantum limit.

One final comment here is that the classical conductivity tensor, in the form shown in Appendix A, and based solely upon considerations of intrinsic symmetry, is well-suited to highly disordered systems in which the influence of crystal symmetry tends to be diminished. Thanks to effective medium theory, composites, with non-conducting phases, do not present serious difficulty in fashioning a conductivity tensor in the manner of (7C), provided that the conducting phase is comfortably above the percolation limit.

Concluding Remarks

In this paper an attempt has been made to demonstrate the Heisenberg Uncertainty Principle as more than just an historical curiosity but rather as the basis of a simple methodology for the determination of certain physical properties at the quantum level in a manner easily understood by engineering undergraduates. The set of examples outlined here is in no way complete [18], implying that various other quantized physical properties can probably be derived by the present methodology simply by focusing on basic definitions and by manipulating the uncertainties in the parameters involved in those definitions into a product characteristic of Heisenberg's original formulation. Given the advent, for example, of cutting-edge electronic devices incorporating nanowires, rapid single flux quantum logic devices, ultra high-density magnetic data storage devices with near atomic-sized elements, it seems reasonable to suggest that even briefly exposing engineering students to the quantized nature of associated basic physical properties is a worthwhile endeavor.

APPENDIX A

Once again, with Eq.(7) as in Eq.(2), the uncertainty in magnetic flux, in an atomic-scale system, may be regarded as the product of the uncertainty in **B** (ΔB) and the uncertainty in the cross- sectional area (ΔS).

i.e. $\qquad\qquad\qquad \Delta\varphi = \Delta B . \Delta S, \qquad (2a)$

The Lorentz force definition may be used, here, to express ΔB as

$$\Delta B = \Delta F/(e\Delta v)$$

where ΔF is the uncertainty in the Lorentz force corresponding the uncertainty in the azimuthal velocity, Δv, of the charge, e. At the same time, uncertainty, ΔS, in the cross-sectional area of the magnetic flux quantum can be related to radial uncertainty, Δr, via $\Delta S = 2\pi r\Delta r$

As a result, Eq.(2a) transforms to

$$\Delta\varphi = (\Delta F/e)(2\pi r/\Delta v)\Delta r$$

$$= (\Delta F\Delta r)\Delta t/e$$

$$= (\Delta E . \Delta t)/e$$

$$= h/e$$

Once again, $\Delta\varphi$ may be treated as a random variable, the lower extremum of which, clearly, is zero. The expectation of $\Delta\varphi$ is, once again, for a symmetric probability density function, of magnitude $E[\Delta\varphi] = h/2e$, in agreement with the accepted value, the inverse Josephson constant[3].

APPENDIX B

In macroscopic systems, magnetic moment, μ, is expresssed as an area integral of a current loop In nanosized systems, it can simply be expressed as $\mu = IS$ *(1b)*

Here, again, as in Eq.(2), the uncertainty in magnetic moment, $\Delta\mu$, may be regarded as the product of the uncertainty in I (ΔI) and the uncertainty in the cross-sectional area (ΔS). As such $\Delta\mu = \Delta I\,\Delta S$. *(2b)*
The uncertainty in current, ΔI may be expressed as $\Delta I = \Delta q/\Delta t$, where Δq is the uncertainty in the passage of charge within the time uncertainty, Δt. Of course, at the nanoscale level of things, this charge uncertainty may be equated to the electronic charge, e. The uncertainty in the cross-sectional area, associated with the quantum of magnetic moment, ΔS, maybe expanded just as before in Appendix A. As a consequence, Eq.(2b) may be re-expressed as $\Delta\mu = (e/\Delta t)(2\pi r\Delta r)$ *(3b)*
Multiplying top and bottom of the right side of Eq. (3b) by the electron rest mass, m_o, and re-grouping, results in
$$\Delta\mu = e\,m_o\,(2\pi r/\Delta t)\Delta r/m_o \qquad\qquad (4b)$$

$$= e\,(m_o\,\Delta v)\Delta r/m_o \qquad\qquad (5b)$$

$$= e\,(\Delta p)(2\pi\Delta r)/2\,\pi m_o \qquad\qquad (6b)$$

$$= e\,(\Delta p\,\Delta x)/2\,\pi\,m_o \qquad\qquad (7b)$$

By invoking the HUP, *Eq.(7b)* reduces to $\Delta\mu = e\,h/2\pi\,m_o$.
In a similar fashion to the approach of Appendix A, the expectation, μ_x , for the quantum of orbital magnetic moment is $\mu_x = e\,h/4\pi m_o$, *(8b)* a result, once again, consistent with the literature [1].

APPENDIX C

Hydrodynamic Model of the General Conductivity Tensor in Semiconductors

The basis of this model, given only in outline, here, is classical dispersion theory with the focus on materials with a single dominant carrier [12] and with the polarization of those carriers and all other bound charges considered separately. The influence of the latter is simply embodied in a non-dispersive (real) component of the (absolute) permittivity, ε_B , associated directly with the lattice, with essentially no impact on the following analysis. The focus is completely on the free carriers, of number density, n, which move with velocity, **v**, when exposed to a local electric field
$$\mathbf{E} = \mathbf{E}_o \exp(-i\omega t)$$
of angular frequency, ω . If a dc external magnetic induction field,
$$\mathbf{B} = [0, 0, B_z]$$
is applied (in the interest of algebraic simplicity, along the z-direction), then the equation of motion is simply

$$\frac{d\mathbf{v}}{dt} + \omega_R\mathbf{v} = -\left(\frac{e}{m^*}\right)(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \qquad\qquad (1C)$$

(for the case where the dominant carrier is the electron), where e is the magnitude of the electronic charge, m^* the cyclotron effective mass and ω_R is an angular carrier relaxation frequency heavily influenced by cyclotron precession. The solution of (1C) has been described in detail previously [20] and is only summarized here. The complex relative permittivity tensor, $[\kappa]$, arrived at is

$$[\kappa]=\begin{bmatrix} \kappa_B - \omega_p{}^2 G & -\omega_p{}^2 H & 0 \\ \omega_p{}^2 H & \kappa_B - \omega_p{}^2 G & 0 \\ 0 & 0 & \kappa_B - \omega_p{}^2 J \end{bmatrix} \qquad (2C)$$

where κ_B is the relative permittivity of bound charge associated with the lattice and

$$\omega_p = \left[\frac{ne^2}{\varepsilon_o m^*}\right]^{1/2}$$

is the (angular) plasma frequency.

Here, $G = \dfrac{A}{A^2 + C^2}$; $H = \dfrac{C}{\left(A^2 + C^2\right)}$; $J = \dfrac{1}{A}$, $\qquad (3C)$

where $A = \omega(\omega + i\omega_R)$, $\omega_R = \dfrac{2\pi}{\tau}$, $C = i\omega\omega_c$, $F = \dfrac{e}{m^*}$ and $\omega_c = FB_z$

is the cyclotron frequency of the carriers in the presence of the applied external magnetic field.

Explicit forms for the leading diagonal elements, ε_{xx}, ε_{yy}, in terms of cyclotron frequency (or of applied magnetic field) are easily obtained from the above tensor equation. Since those elements are complex, the imaginary parts (multiplied by $-i\omega\varepsilon_o$) lead directly to a corresponding conductivity tensor. For example

$$\varepsilon_{xx} = \kappa_B - \omega_p{}^2 G = \kappa_B - \frac{\omega_p{}^2 A}{\left(A^2 + C^2\right)}. \qquad (4C)$$

It is a simple matter to express (4A) in terms of its real and imaginary parts as follows

$$\varepsilon_{xx} = \varepsilon_{xxr} + i\varepsilon_{xxi}$$

where

$$\varepsilon_{xxi} = \varepsilon_{yyi} = \frac{\omega_p{}^2 \left[2\omega^2\omega_R - \omega_R\left\{\omega^2 - \left(\omega_c{}^2 + \omega_R{}^2\right)\right\}\right]}{\omega\left[\left\{\omega^2 - \left(\omega_c{}^2 + \omega_R{}^2\right)\right\}^2 - 4\omega^2\omega_R{}^2\right]} \qquad (5C)$$

By similar reasoning, the off-diagonal elements of the above tensor can be similarly expressed, this time as an explicit first-order function of ω_c.

i.e. $\qquad \varepsilon_{xy} = -\varepsilon_{yx} = \dfrac{-i\omega_p{}^2\omega_c}{\omega\left[\left(\omega + i\omega_R\right)^2 - \omega_c{}^2\right]} \qquad (6C)$

From (2C), (5C) and (6C), it is a relatively simple matter to construct a zero-frequency optical conductivity tensor, [σ]

$$[\sigma]=\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ -\sigma_{xy} & \sigma_{xx} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix} \qquad (7C)$$

where

$$\sigma_{xx} = \frac{\varepsilon_o\omega_p{}^2\omega_R}{\left(\omega_R{}^2 + \omega_c{}^2\right)} \qquad (8C)$$

$$\sigma_{zz} = \frac{\varepsilon_o \omega_p{}^2}{\omega_R}, \qquad\qquad (9C)$$

and
$$\sigma_{xy} = \frac{\varepsilon_o \omega_p{}^2 \omega_c}{\omega_R{}^2 + \omega_c{}^2}, \qquad (10C)$$

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